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Inverse Scattering and Soliton Solutions for a New Integrable System

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A new integrable system describing the process of self-induced transparency with spatial dispersion is analyzed from the viewpoint of inverse scattering transform. The coupled set of Gelfand–Levitan equation is established and solved for one soliton solutions. The explicit structure of the solutions are exhibited graphically.

KEY WORDS: IST; soliton; new integrable system; Gelfand–Levitan equation.

1. INTRODUCTION

One of the most important application of solitary waves is in the domain of nonlinear optics (Agarwal, 2001). The whole subject of nonlinear optics had its genesis in the study of McCall and Hahn (1967) who were the pioneers in this domain. Later various other regions of subject were explored by other authors (Hasegawa, 1998). Of these, the most widely studied system is the Nonlinear Schrodinger Equation (Kivshar and Malomed, 1989) that describes a host of phenomena. Of course, exhaustive work has been done by Maimitsov *et al.* (1990). Here, in this work, we have examined a new set of integrable nonlinear equations describing the phenomenon of self-induced transparency with spatial dispersion. These equations were initially written by Agranovich and Rupasov (1976), but its analysis from the point of view of IST was not taken up. Here we have analyzed the Lax pair of the system, and have set up the Gelfand–Levitan–Marchenko (Ablowitz and Clarkson, 1991) equation and have obtained the solitary wave solution explicitly.

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2. FORMULATION

The equation under consideration can be written as

$$\begin{aligned}
\frac{\partial a_1}{\partial t} &= \frac{\beta}{16}\,(r_x q_{xxx} - q_x r_{xxx}) + \frac{\alpha}{2}\,(r q_x + q r_x) \\
q_{xt} &= \frac{\beta}{2}(q_{xxx} a_1 - q_x a_{1_{xxx}}) + 4\alpha q a_1 \\
r_{xt} &= -\frac{\beta}{2}(r_{xxx} a_1 - r_x a_{1_{xx}}) + 4\alpha r a_1
\end{aligned} \tag{1}$$

where the variables a_1, q, and r, respectively denote the density, electric field, and polarization inside the medium. These equations have been seen to be associated with the following Lax pair,

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi \tag{2}$$

where,

$$U = ik\begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}, \quad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{3}$$

with A, B, and C, given as

$$\begin{aligned}
A &= \frac{\alpha}{ik} - \beta a_1 k^2 - \frac{\beta ik}{8}(q_x r_{xx} - r_x q_{xx}) \\
B &= \alpha q + \frac{\beta q_x k^2}{2} - \frac{\beta ik}{4}(a_1 q_{xx} - q_x a_{1_x}) \\
C &= \alpha r - \frac{\beta r_x k^2}{2} - \frac{\beta ik}{4}(a_1 r_{xx} - r_x a_{1_x}) \\
2b_1 &= -q_x, \quad 2c_1 = r_x
\end{aligned} \tag{4}$$

For the construction of soliton like solution with the boundary condition, $|x| \to \infty$, we assume that asymptotically, $a_1 \to 1$, $b_1, c_1 \to 0$, so that the first equation of (2) reduces to

$$\Psi_x = ik\sigma_z\Psi \tag{5}$$

σ_z being the Pauli matrix, whence we get

$$\Psi = A\,e^{ik\sigma_z x} \tag{6}$$

with A any constant matrix. Now taking recourse to the method of variation of parameters, we get the integral equation equivalent to the original Lax space part

$$\Psi(x) = \exp[ik\sigma_z(x - x_0)]\,\Psi(x_0) + ik\int_{x_0}^{x} \exp[ik\sigma_z(x - y)]\,[S(y) - \sigma_z]\,\Psi(y)\,dy \tag{7}$$

where x_0 denotes any initial point on the x axis. Now the Jost functions $F(x, k)$, $G(x, k)$ are defined, respectively, by the boundary conditions

$$\begin{aligned}\Psi(x, k) &\to \exp(ik\sigma_z x) \ as \ x \to \infty \\ \varphi(x, k) &\to \exp(ik\sigma_z x) \ as \ x \to -\infty \end{aligned} \tag{8}$$

so that we get

$$\begin{aligned}\Psi(x, k) &= \exp(ik\sigma_z x) - ik \int_x^{\infty} \exp[ik\sigma_z(x - y)]\, [S(y) - \sigma_z)]\, \Psi(y, k)\, dy \\ \varphi(x, k) &= \exp(ik\sigma_z x) + ik \int_{-\infty}^{x} \exp[ik\sigma_z(x - y)]\, [S(y) - \sigma_z]\, \varphi(y, k)\, dy \end{aligned} \tag{9}$$

Being dictated by these forms of integral equations we define two kernels $K(x, y)$ and $N(x, y)$ as follows;

$$\begin{aligned}\Psi(x, k) &= \exp(ik\sigma_z x) - ik \int_x^{\infty} K(x, k) \exp(ik\sigma_z y)\ dy \\ \varphi(x, k) &= \exp(ik\sigma_z x) + ik \int_{-\infty}^{x} N(x, y) \exp(ik\sigma_z y)\ dy \end{aligned} \tag{10}$$

Now demanding that these are solutions of the Lax equation (2) we get

$$\begin{aligned}\sigma_z + K(x, x) - S(x) - S(x)K(x, x)\sigma_z &= 0 \\ \frac{\partial K}{\partial x}\sigma_z + S(x)\frac{\partial K}{\partial y}\sigma_z &= 0 \\ \sigma_z + N(x, x) - S(x) - S(x)N(x, x)\sigma_z &= 0 \\ \frac{\partial N}{\partial x}\sigma_z + S(x)\frac{\partial N}{\partial y}\sigma_z &= 0 \end{aligned} \tag{11}$$

3. TIME EVOLUTION OF DATA

For the complete solution of the nonlinear problem, it is necessary that the time evolution of the scattering data is determined by the second equation of (2). It is easily observed that in the asymptotic limit described before

$$\Psi(x, t) \simeq G(x, k)A(t). \tag{12}$$

with

$$A(t) = \exp\left[-\left(\frac{i\alpha}{k} + \beta k^2\right)\sigma_z t\right] A(0) \tag{13}$$

whereas on the other side of the x - axis $\Psi(x, t)$ is written as

$$\Psi(x, t) = F(x, k)T(k, t)A(t) \tag{14}$$

along with the stipulation that

$$T(k, t) = R(t)T(k, 0)R(t)^{-1} \tag{15}$$

with

$$R(t) = \exp\left[-\left(\frac{i\alpha}{k} + \beta k^2\right)\sigma_z t\right] \tag{16}$$

Rewriting $T(k, t)$ as

$$T(k, t) = \begin{pmatrix} a(k, t) & -\bar{b}(k, t) \\ b(k, t) & \bar{a}(k, t) \end{pmatrix} \tag{17}$$

Eq. (16) implies

$$\begin{aligned} a(k, t) &= a(k, 0) \\ \bar{b}(k, t) &= \bar{b}(k, 0)\exp\left[-2\left(\frac{i\alpha}{k} + \beta x^2\right)t\right] \end{aligned} \tag{18}$$

4. GELFAND–LEVITAN EQUATION

The derivation of GLM equation starts from the basic linear relation

$$\begin{aligned} \Psi &= a(k)\bar{\phi} + b(k)\phi \\ \bar{\Psi} &= -\bar{a}(k)\phi + \bar{b}(k)\bar{\phi} \end{aligned} \tag{19}$$

For convenience we write out the first component of (19) and divide by a (k) to get

$$\frac{\Psi_1}{a(k)} = \bar{\phi}_1 + \frac{b(k)}{a(k)}\phi_1 \tag{20}$$

Substituting the integral representation of the Jost functions, and taking fourier transform, one gets the required GLM equations. Since the procedure to be followed is a routine one, we omit the derivation and only quote the final result. The four equations are

$$iK_{11}(x, z) - i\int_x^\infty \frac{b}{a'}K_{12}(x, y)\exp\left[-ik(y + z)\right]dy = 0 \tag{21}$$

$$\begin{aligned} iK_{21}(x, z) &+ \int_x^\infty \frac{b}{a'}K_{22}(x, y)\exp\left[-ik(z + y)\right]dy \\ &+ \frac{ib}{a'k}\exp[-ik(z + x)] = 0 \end{aligned} \tag{22}$$

$$iK_{12}(x, z) - \frac{i\bar{b}}{\bar{a}'k}\exp\left[i\bar{k}(x + z)\right] - \int_x^\infty K_{11}\frac{\bar{b}}{a'}\exp\left[i\bar{k}(z + y)\right]dy = 0 \tag{23}$$

$$iK_{22}(x,z) - \int^{\infty} K_{21}(x,y)\frac{\bar{b}}{a'}\exp\left[i\bar{k}(y+z)\right]dy = 0 \tag{24}$$

To solve this system of integral equations we set

$$K_{11}(x,z) = CL(x,t)\,e^{-ikz} \tag{25}$$

with $c = \frac{b}{a'}$, whence Eq. (22) yeilds,

$$K_{12}(x,z) = \frac{\bar{c}}{k}\exp\left[i\bar{k}(x+z)\right] - \frac{c\bar{c}\exp\left[i\bar{k}z\right]\exp[-i(k-\bar{k})x]}{k-k}L(x,t) \tag{26}$$

we also get from (23)

$$L(x,t) = i\int_x^{\infty} K_{12}(x,y)\,e^{-iky}\,dy \tag{27}$$

whence we get

$$L(x,t) = \frac{\bar{c}(k-\bar{k})\,e^{-ikx}\,e^{2i\bar{k}z}}{\bar{k}A} \tag{28}$$

with $A = \left(k-\bar{k}\right)^2 + c\bar{c}e^{-2i(k-\bar{k})x}$ which in turn yields,

$$K_{11} = \frac{c\bar{c}(k-\bar{k})\,e^{-ik(x+z)}\,e^{2i\bar{k}x}}{\bar{k}A} \tag{29}$$

$$K_{12} = \frac{\bar{c}}{k}e^{i\bar{k}(x+z)} - \frac{c\bar{c}^2\,e^{i\bar{k}(x+z)}\,e^{-2i(k-\bar{k})x}}{\bar{k}A} \tag{30}$$

And in a similar fashion

$$K_{22} = \frac{c\bar{c}(k-\bar{k})\,e^{ik(x+z)}\,e^{-2ikx}}{kA} \tag{31}$$

$$K_{21} = -\frac{c}{k}\,e^{-ik(x+z)} + \frac{c^2\bar{c}\,e^{-ik(x+z)}\,e^{-2i(k-\bar{k})x}}{kA} \tag{32}$$

The form of the nonlinear fields can now be explicitly obtained using Eq. (11). We get

$$a_1 = \frac{(16\eta^4 + c^2\bar{c}^2\,e^{8\eta x}) + 24\eta^2 c\bar{c}\,e^{4\eta x}}{(16\eta^4 + c^2\bar{c}^2\,e^{8\eta x}) - 8\eta^2 c\bar{c}\,e^{4\eta x}}$$

$$b_1 = 8i\bar{c}\eta\frac{e^{2\eta x}[4\eta^2 + c\bar{c}\,e^{4\eta x}]}{(16\eta^4 + c^2\bar{c}^{-2}\,e^{8\eta x}) - 8\eta^2 c\bar{c}\,e^{4\eta x}}$$

$$c_1 = 8ic\eta\frac{e^{2\eta x}[4\eta^2 + c\bar{c}\,e^{4\eta x}]}{(16\eta^4 + c^2\bar{c}^{-2}\,e^{8\eta x}) - 8\eta^2 c\bar{c}\,e^{4\eta x}}$$

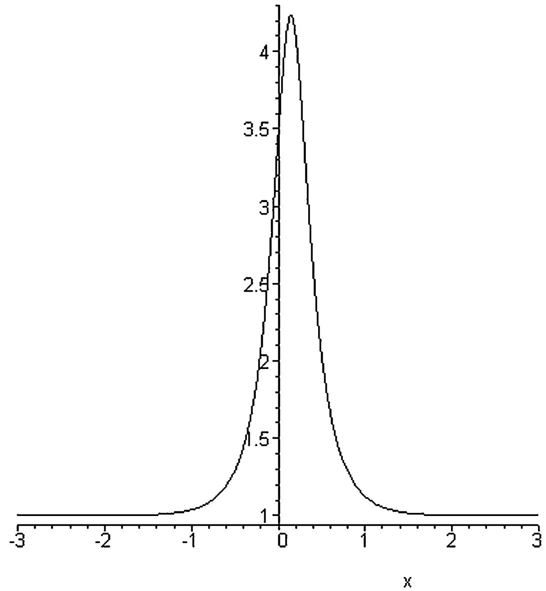

Fig. 1. Stationary form of (a_1).

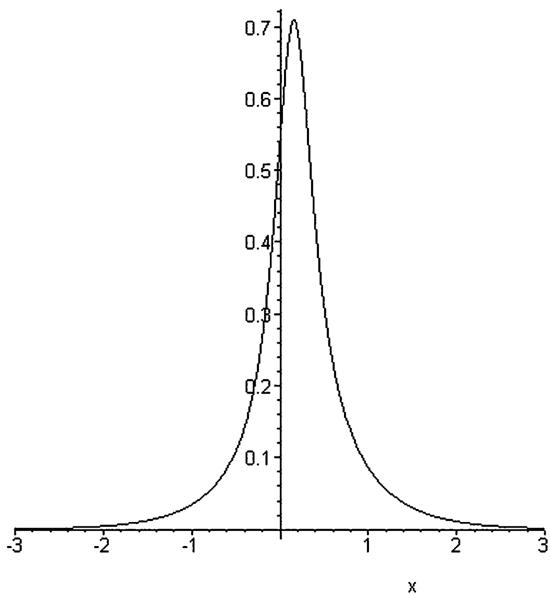

Fig. 2. Stationary form of (b_1).

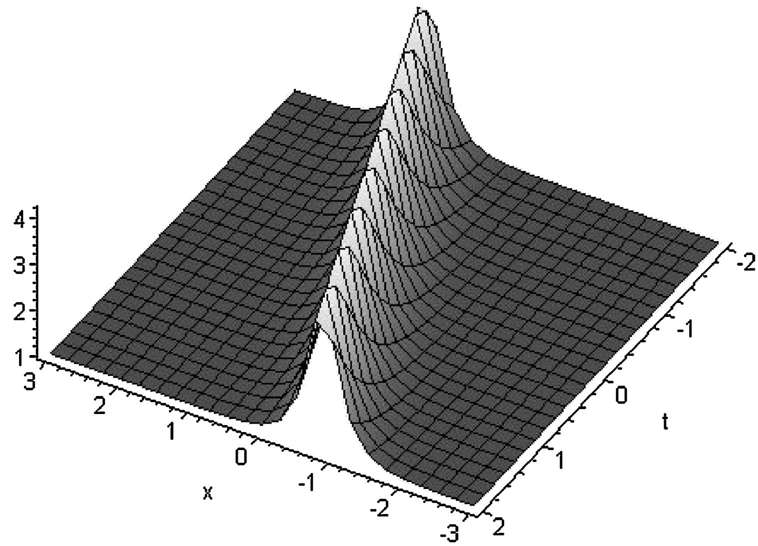

Fig. 3. Three-dimensional time evolution of a_1.

where we have set $k = i\eta$, $\overline{k} = -i\eta$. In the above expressions c, $\overline{c}$, are complex functions of time. To ascertain the form of these excitations we have plotted the stationary form of a_1, b_1 in Figs. 1 and 2, which indicate perfect solitary structure, whereas Fig. 3 and 4 show their time evolution in a 3D0 graphical representation.

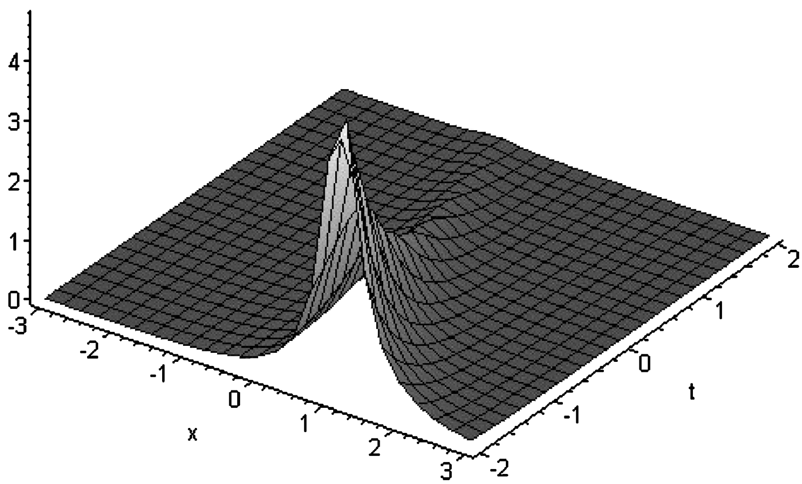

Fig. 4. Three-dimensional time evolution of b_1.

5. CONCLUSION

In our above analysis we have obtained explicit solitonic solutions for a new set of coupled integrable system, which is very important in the domain of nonlinear optics.

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